Signal processing

Prof. L. Vandendorpe

UCL Communications and Remote Sensing Lab.

UCL
Université catholique de Louvain

Pros associated with discrete time signals

- $+$: Flexibility
- $+$: Adaptability (mobile comm, compression)
- $+$: Accuracy and reproducibility
- $+$: Functionalities
  - compression
  - error correction coding
  - information protection

Cons associated with discrete time signals

- - : Limitation due to speed@accuracy of ADC
- - : Quantization : non linear phenomena

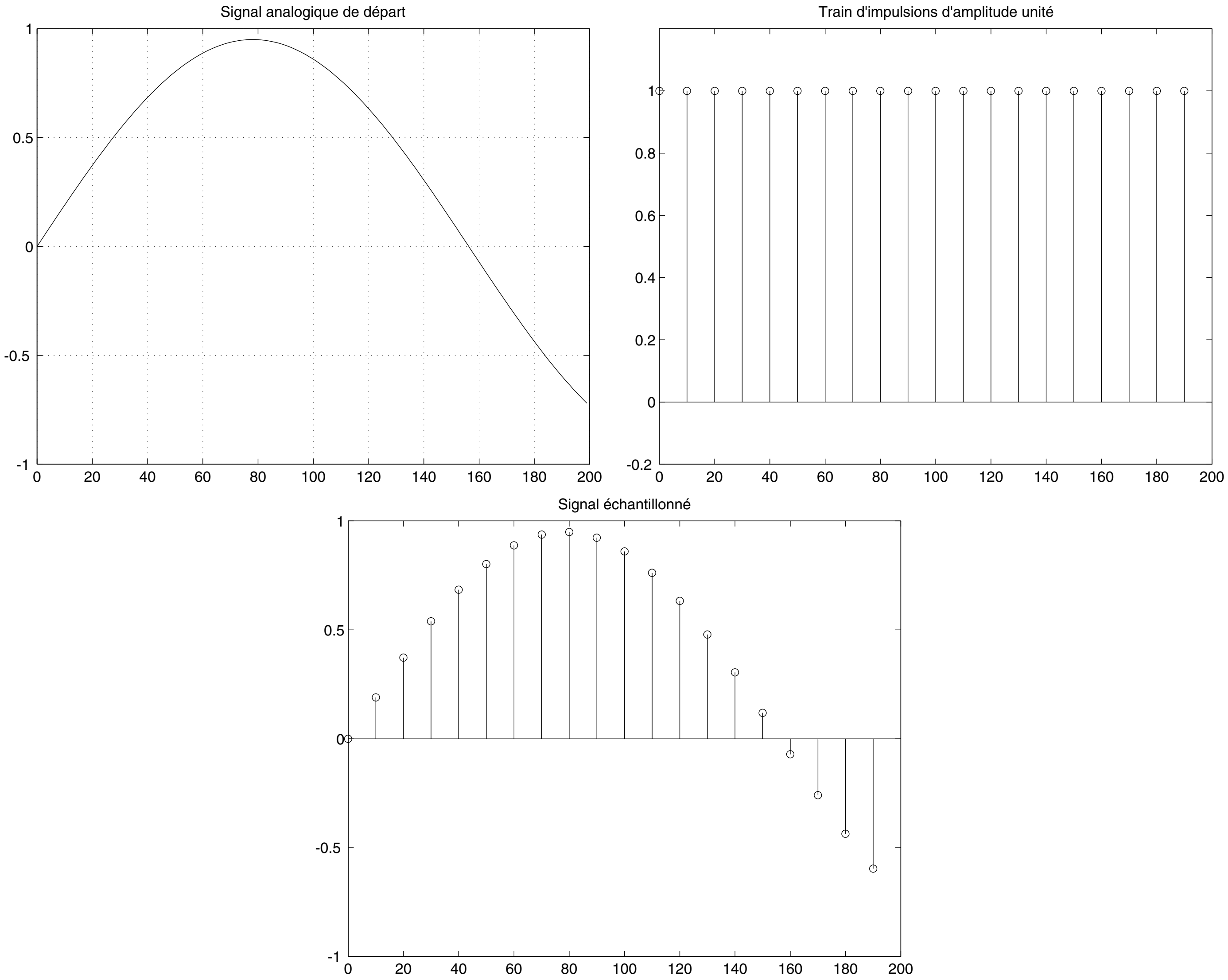
Words

- Sampling : discrete time signals
- Digital : finite accuracy of samples (quantization)

Sampling (1/2)

- Take samples every T seconds (sampling period)
- $f_e = 1/T$: sampling frequency
- Definition of the sampled signal $x_e(t)$ from the input signal $x_a(t)$
- Let $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- $x_e(t) = x_a(t)\, p(t) = x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_a(nT)\, \delta(t - nT)$

Sampling (2/2)


Signal analogique de départ
1
0.5
0
-0.5
-1
0
20
40
60
80
100
120
140
160
180
200
Train d'impulsions d'amplitude unité
1
0.8
0.6
0.4
0.2
0
-0.2
0
20
40
60
80
100
120
140
160
180
200
Signal échantillonné
1
0.5
0
-0.5
-1
0
20
40
60
80
100
120
140
160
180
200

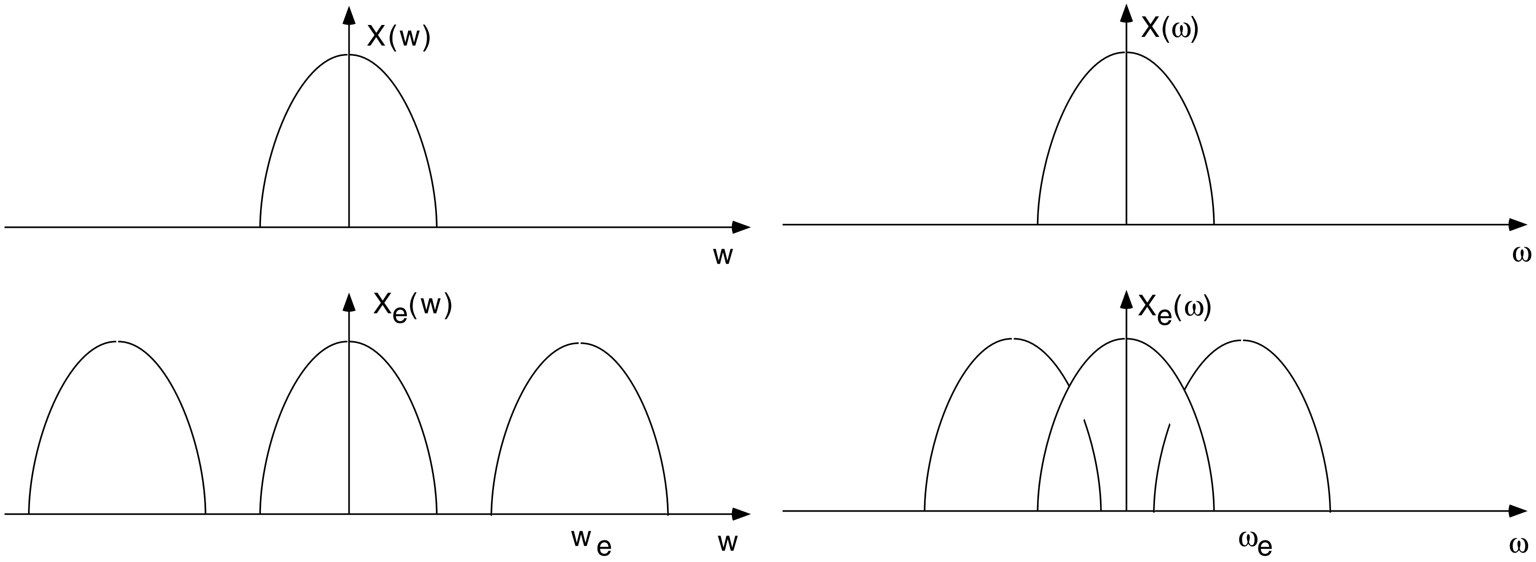
Sampling theorem (1/3)

- Fourier series expansion of $p(t)$: $p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{2\pi jkt/T}$
- Spectrum : $P(\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k/T)$
- About the sampled signal : $X_e(\omega) = \frac{1}{2\pi} X_a(\omega) \otimes P(\omega)$
- Result : $X_e(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\omega - 2\pi k/T)$

Sampling theorem (2/3)

- Main effect : spectrum repetition
- Condition for no aliasing : $f_e \geq 2 f_{max}$ (Shannon theorem)
- If not fulfilled : prefiltering is required
- Even if fulfilled : prefiltering to avoid noise aliasing

Sampling theorem (3/3)



Signal recovery (1/2)

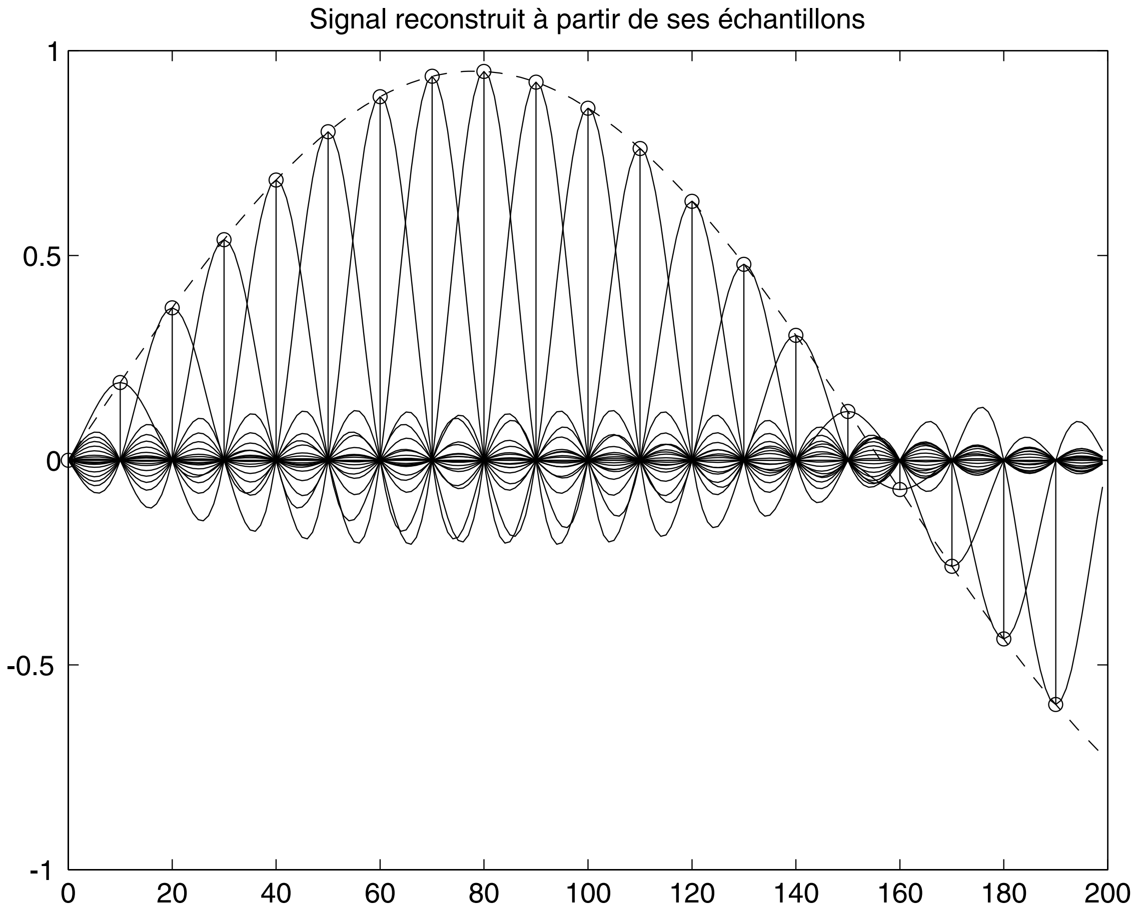
- Remove repeted spectra $(*T)$

- Use filter

$$H(\omega) = \begin{cases} \text{T} & -\pi/T \le \omega \le \pi/T \\ 0 & \text{otherwise} \end{cases}$$

- Impulse response $h(t) = \dfrac{sin[\pi t/T]}{\pi t/T}$

Signal recovery (2/2)



Conclusions

- Sampling theorem : sufficient but not necessary
- Example : bandpass signals

Sequence (1/2)

- Definition-correspondance : $x(n) = x_a(nT)$

- Spectrum definition

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jn\Omega} \tag{1}$$

- z-transform definition

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \tag{2}$$

Sequence (2/2)

- Spectra

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{jn\Omega}\, \mathrm{d}\Omega \\ &= x_a(nT) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\omega)\, e^{j\omega nT}\, \mathrm{d}\omega \\ &= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} X_a(\omega - 2\pi k/T)\, e^{jn\Omega}\, \mathrm{d}\Omega/T \qquad (3) \end{aligned}$$

- Hence

$$X(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a(\omega - 2\pi k/T) \qquad (4)$$

Correspondance

- Frequencies

$$F = f/f_e = \omega/\omega_e = fT \tag{5}$$

$$\Omega = 2\pi F = \omega T = 2\pi f T \tag{6}$$

Sampling rate conversion (1/3)

- One goes from $x(n)$ ($x_a(t)$ sampled with T) to $y(m)$ ($x_a(t)$ sampled with T')

- Analog solution
  - from $x(n)$ build the analog version
  - if needed filter the analog version (sampling theorem)
  - resample with T'

- see figure

Sampling rate conversion (2/3)

- Assume interpolation with $h(t)$

- Values which matter : T spaced sampling of $h(t)$

- Impulse response and number of elements depends on T' and m

$$y(m) = \sum_{n=-N_1}^{N_2} \widehat{h}(mT' - nT)\, x(n) \tag{7}$$

- see example for $T = 2T'$

Sampling rate conversion (3/3)

- Often

$$\frac{T'}{T} = \frac{M}{L} \tag{8}$$

- L fold upsampling and M fold dowsampling (the order matters)
- L impulse responses are needed
- Hence system linear
- But no longer time invariant

M fold downsampling

- Sampling rate is reduced or $T' = MT$
- Take care of aliasing
- First filter $x(n)$ by means of

$$H(e^{j\Omega}) = \begin{cases} 1 & |\Omega| \le \pi/M \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

- Only take (compute) every Mth sample

M fold downsampling

- Output of filter $w(n)$

- Downsampled signal

$$y(m) = w(mM) = \sum_{n=-\infty}^{\infty} h(n)\,x(mM-n) = \sum_{n=-\infty}^{\infty} h(mM-n)\,x(n) \tag{10}$$

- Only one impulse response is needed

- However downsampling still not time invariant process : see impulse

M fold downsampling: spectrum

- Use analog formalism

- Signal filtered with $h(t)$

$$w_e(t) = \sum_{n=-\infty}^{\infty} w_a(nT)\,\delta(t-nT) \tag{11}$$

- Spectrum

$$W_e(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} W_a(\omega - 2\pi k/T) \tag{12}$$

M fold downsampling: spectrum

- Downsampled signal

$$y_e(t) = \sum_{n=-\infty}^{\infty} w_a(nMT)\,\delta(t - nMT) \tag{13}$$

- Spectrum

$$Y_e(\omega) = \frac{1}{MT} \sum_{k=-\infty}^{\infty} W_a(\omega - 2\pi k/MT) \tag{14}$$

$$Y_e(\omega) = \frac{1}{MT} \sum_{k=-\infty}^{\infty} W_a(\omega - 2\pi k/MT) = \frac{1}{M} \sum_{k=0}^{M-1} W_e(\omega - 2\pi k/MT) \tag{15}$$

- M times faster spectrum repetition

M fold downsampling: digital implementation

- Filter

$$y(m) = w(mM) = \sum_{n=-\infty}^{\infty} h(mM-n)\,x(n) \tag{16}$$

- About spectra

$$\begin{aligned} y(m) &= w(mM) \\ &= \frac{1}{2\pi}\int_0^{2\pi} Y(e^{j\Omega'})\,e^{jn\Omega'}\,\mathrm{d}\Omega' \\ &= \frac{1}{2\pi}\int_0^{2\pi} W(e^{j\Omega})\,e^{jnM\Omega}\,\mathrm{d}\Omega \end{aligned} \tag{17}$$

M fold downsampling: digital implementation

- Manipulation

$$w(mM) = \frac{1}{2\pi} \sum_{k=0}^{M-1} \int_{2\pi k/M}^{2\pi(k+1)/M} W(e^{j\Omega})\, e^{jnM\Omega}\, \mathrm{d}\Omega \tag{18}$$

- Define $\Omega'' = \Omega - 2\pi k/M$ and $k' = M - k$.

$$w(mM) = \frac{1}{2\pi} \int_{0}^{2\pi/M} \sum_{k=0}^{M-1} W\left[e^{j(\Omega'' - 2\pi k/M)}\right] e^{jnM\Omega''}\, \mathrm{d}\Omega'' \tag{19}$$

- Define $\Omega' = M\Omega''$

$$w(mM) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{M} \sum_{k=0}^{M-1} W\left[e^{j(\Omega' - 2\pi k)/M}\right] e^{jn\Omega'}\, \mathrm{d}\Omega' \tag{20}$$

M fold downsampling: digital implementation

- Result

$$Y(e^{j\Omega}) = \frac{1}{M} \sum_{k=0}^{M-1} W\left[e^{j(\Omega-2\pi k)/M}\right] \tag{21}$$

- M times faster repetition
- Additional scale change
- see example

L fold upsampling

- Sampling rate is increased or $T' = T/L$
- Repetition L times less
- Hence eliminate some repeated version $(L-1$ out of $L)$
- First insert zeroes $(L-1$ between samples) to increase working rate
- Then apply

$$H(e^{j\Omega}) = \begin{cases} L & |\Omega| \leq \pi/L \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

L fold upsampling

- Analog view

$$X_{e,T}(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a(\omega - 2\pi k/T) \tag{23}$$

$$\begin{aligned} X_{e,T'}(\omega) &= \frac{1}{T'} \sum_{k=-\infty}^{\infty} X_a(\omega - 2\pi k/T') \\ &= \frac{L}{T} \sum_{k=-\infty}^{\infty} X_a(\omega - 2\pi kL/T) \end{aligned} \tag{24}$$

L fold upsampling

- Zero insertion

$$w(m) = \begin{cases} x(n) & \text{si } m = nL \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

$$\begin{aligned} W(e^{j\Omega}) &= \sum_{m=-\infty}^{\infty} w(m)e^{-jm\Omega} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-jnL\Omega} \\ &= X(e^{jL\Omega}) \end{aligned} \tag{26}$$

- Does not modify spectrum

- Remark : lost spectrum can never be recovered-see example

M/L conversion

- Definition

$$\frac{T'}{T} = \frac{M}{L} \tag{27}$$

- L fold upsampling THEN M fold downsampling

- Otherwise filter too much

- See block diagram

M/L conversion

- Mix filters together

- Hence

$$H(e^{j\Omega}) = \begin{cases} L & |\Omega| \leq \ \min(\pi/L, \pi/M) \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

- If $x(k)$ at the input, $s(n)$ after filter, $y(m)$ at the output

$$S(e^{j\Omega}) = H_1(e^{j\Omega})\, X(e^{jL\Omega}) \tag{29}$$

$$\begin{aligned} Y(e^{j\Omega}) &= \frac{1}{M} \sum_{k=0}^{M-1} H_2 \left[e^{j(\Omega-2\pi k)/M}\right] S \left[e^{j(\Omega-2\pi k)/M}\right] \\ &= \frac{1}{M} \sum_{k=0}^{M-1} H \left[e^{j(\Omega-2\pi k)/M}\right] X \left[e^{j(L\Omega-2\pi k)/M}\right] \end{aligned} \tag{30}$$

M/L conversion

- With an ideal filter $H = H_1 H_2$

$$Y(e^{j\Omega}) = \begin{cases} \frac{L}{M} X(e^{j\Omega L/M}) & |\Omega| \leq \min(\pi, \pi L/M) \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

Bandpass signals

- Up to now : lowpass signals hence lowpass processing filters
- Here bandpass signals
- If naive application of sampling theorem : see example
- Better : move spectrum close to origin
- Two methods
  - sampling rate modulation
  - quadrature modulation or complex envelope

Sampling rate modulation

- Assume a digital signal
- Isolate band of interest by means of $h_{bp}(n)$
- If M fold downsampling repetition around $2\pi i/M$
- No aliasing if

$$H_{bp}(e^{j\Omega}) = \begin{cases} 1 & k\pi/M \leq |\Omega| \leq (k+1)\pi/M \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

Sampling rate modulation

- The decimator produces the shift on the frequency axis
- One version falls around the origin
- see example
- Depending on k there may be an inversion - compensated by $(-1)^n$

Sampling rate modulation : reverse steps

- Compensate the $(-1)^n$ if operated
- Insert $L-1$ zeroes
- Bandpass filter
- New position may be different from old one

Quadrature modulation

- Previous method limited
- Assume BW $\Omega_\triangle$ around Ω_0
- Real signal = redundancy in the spectrum
- Even real part, odd imaginary part
- Information for positive (or negative) frequencies is sufficient

Quadrature modulation

- Keep positive frequencies
- Move around origin
- One gets a complex signal in general : the complex envelope $e_1(n)$
- One defines

$$\begin{aligned} e_1(n) &= 2\left[x(n)e^{-j\Omega_0 n}\right] \otimes h(n) \\ &= 2\left[x(n)\cos(\Omega_0 n)\right] \otimes h(n) - j2\left[x(n)\sin(\Omega_0 n)\right] \otimes h(n) \\ &= x_1(n) - jx_2(n) \qquad (33) \end{aligned}$$

Quadrature modulation

- Real sequences x_1 and x_2 are the I-Q components or Rice components
- They have BW Ω_Δ and can be downsampled
- One has

$$E_1(e^{j\Omega}) = 2\,H(e^{j\Omega})X\left[e^{j(\Omega+\Omega_0)}\right] \qquad (34)$$

$$H(e^{j\Omega}) = \begin{cases} 1 & 0 \leq |\Omega| \leq \Omega_\Delta/2 \\ 0 & \text{otherwise} \end{cases} \qquad (35)$$

Quadrature modulation

- Negative frequencies could be used instead

$$\begin{aligned} e_2(n) &= 2\left[x(n)e^{j\Omega_0 n}\right] \otimes h(n) \\ &= 2\left[x(n)\cos(\Omega_0 n)\right] \otimes h(n) + j2\left[x(n)\sin(\Omega_0 n)\right] \otimes h(n) \\ &= x_1(n) + jx_2(n) \end{aligned} \tag{36}$$

Analytic signal

- Keep positive frequencies only of input signal

$$Z(e^{j\Omega}) = \begin{cases} 2X(e^{j\Omega}) & 0 \leq \Omega \leq \pi \\ 0 & -\pi \leq \Omega \leq 0 \end{cases} \tag{37}$$

$$Z(e^{j\Omega}) = E_1\left[e^{j(\Omega-\Omega_0)}\right] \tag{38}$$

- Quadrature signal or Hilbert transform

$$\begin{aligned} \hat{X}(e^{j\Omega}) &= \left[Z(e^{j\Omega}) - X(e^{j\Omega})\right]/j \\ &= \begin{cases} -j\,X(e^{j\Omega}) & 0 \leq \Omega \leq \pi \\ j\,X(e^{j\Omega}) & -\pi \leq \Omega \leq 0 \end{cases} \end{aligned} \tag{39}$$

- Finally

$$\hat{X}(e^{j\Omega}) = -j\,\mathrm{sign}(\Omega)\,X(e^{j\Omega}) \text{ for } |\Omega| \leq \pi \tag{40}$$

Signal reconstruction

- Reposition $E_1(\omega)$ and $E_2(\omega)$ correctly

$$\begin{aligned} x(n) &= \left[e_1(n)e^{j\Omega_0 n} + e_2(n)e^{-j\Omega_0 n}\right]/2 \\ &= x_1(n)\cos(\Omega_0 n) + x_2(n)\sin(\Omega_0 n) \end{aligned} \tag{41}$$

- see figure

Random signals

- $X(n)$ is random sequence

- Mean

$$\begin{aligned} m_X(n) &= \int_x x\, p_X(x,n)\, \mathrm{d}x && (42) \\ \sigma_X^2(n) &= \int (x - m_X)^2\, p_X(x,n)\, \mathrm{d}x && (43) \end{aligned}$$

- Covariance function

$$\begin{aligned} C_X(n_1,n_2) &= \mathbf{E}\{[X(n_1) - m_X(n_1)]\,[X(n_2) - m_X(n_2)]^*\} \\ &= \int_{x_1}\int_{x_2} [x(n_1) - m_X(n_1)]\,[x(n_2) - m_X(n_2)]^* \\ &\times\ p_X(x_1,n_1,x_2,n_2)\, \mathrm{d}x_1 \mathrm{d}x_2 \end{aligned} \qquad (44)$$

Wide sense stationary (WSS)

- Constant mean $m_X(n) = m_X$

- About covariance

$$\begin{aligned} C_X(n_1, n_2) &= C_X(n = n_1 - n_2) && (45) \\ \sigma_X^2(n) &= \sigma_X^2 && (46) \end{aligned}$$

- Power spectrum: Fourier transformation of covariance

$$S_X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} C_X(n)\, e^{-jn\Omega} \tag{47}$$

- Ergodicity

Sequence filtering

- $X(n)$, $Y(n)$ WSS random, $g(n)$ known impulse response

$$Y(k) = \sum_{m=-\infty}^{\infty} g(m)\, X(k-m) \tag{48}$$

- Covariance

$$C_Y(n) = \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} g(m)\, g(m')\, C_X(n-m+m') \tag{49}$$

- Hence stationarity is preserved

- In other words

$$C_Y(n) = C_X(n) \otimes C_g(n) \tag{50}$$

Sequence filtering

- Output power spectrum

$$\begin{aligned}
S_Y(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} C_Y(n)\, e^{-jn\Omega} \\
&= \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} g(m)\, g(m') \sum_{n=-\infty}^{\infty} C_X(n-m+m')\, e^{-jn\Omega} \\
&= \sum_{m=-\infty}^{\infty} g(m)\, e^{-jm\Omega} \sum_{m'=-\infty}^{\infty} g(m)\, e^{jm'\Omega} \\
&\times \sum_{n=-\infty}^{\infty} C_X(n-m+m')\, e^{-j(n-m+m')\Omega} \\
&= G\left(e^{j\Omega}\right)\, G\left(e^{-j\Omega}\right)\, S_X(e^{j\Omega}) \qquad (51)
\end{aligned}$$

Sequence downsampling

- Assume downsampling phase M_0

$$Y(n) = W(nM + M_0) \tag{52}$$

- About covariance

$$\begin{aligned} C_Y(n) &= \mathbf{E}\{W_c[(n+k)M + M_0]\, W_c[kM + M_0]\} \\ &= C_W(nM) \\ &= C_X(nM) \otimes C_h(nM) \end{aligned} \tag{53}$$

- Still stationary

- About power spectrum

$$S_Y(e^{j\Omega}) = \frac{1}{M} \sum_{k=0}^{M-1} |H\left[e^{j(\Omega-2\pi k)/M}\right]|^2 S_X\left[e^{j(\Omega-2\pi k)/M}\right] \tag{54}$$

Sequence interpolation

- Take care of stationarity

- Interpolation defined as follows

$$W(m+L_0) = \begin{cases} X(n) & m = nL \\ 0 & \text{sinon} \end{cases} \tag{55}$$

- L_0 assumed to be random variable $\in [0, L-1]$

- After filtering

$$\begin{aligned} Y(m) &= \sum_{k=-\infty}^{\infty} g(m-k)\,W(k) \\ &= \sum_{l=-\infty}^{\infty} g(m-lL-L_0)\,X(l) \end{aligned} \tag{56}$$

Sequence interpolation

- Assuming L_0

$$C_Y(n_1, n_2|L_0) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} g(n_1 - m_1 L - L_0)\, g(n_2 - m_2 L - L_0)\, C_X(m_1 - \tag{57}$$

- Average over L_0 (define $u = m_2 - m_1$)

$$C_Y(v) = \frac{1}{L} \sum_{q=-\infty}^{\infty} \sum_{u=-\infty}^{\infty} g(q)\, g(q + v - uL)\, C_X(u) \tag{58}$$

- About power spectrum

$$S_Y(e^{j\Omega}) = \frac{1}{L}\, |G(e^{j\Omega})|^2\, S_X(e^{jL\Omega}) \tag{59}$$

Structures for sampling rate conversion

- Need for efficient structures
- Upsampling : filters a lot of "0"
- Downsampling : computes many samples discarded afterwards
- Tools : graphs, nodes, branches
- See elementary branches

Commutation rules

- No problem for linear time invariant systems

- Care for multirate operations

Transposition and duality

- Transpose structures correspond to complementary operations
- Example : downsampling and upsampling
- For time invariant systems role not modified if
  - branches are inverted
  - inputs and outputs are exchanged
- see examples
- Not the same for time variant systems

Structures for time invariant systems

- Will be investigated for FIR filters
- Direct and direct transpose structure
- Polyphase structures

Direct and direct transpose for FIR

- Convolution

$$y(n) = \sum_{k=0}^{N-1} g(k)\, x(n-k) \tag{60}$$

- see figure with direct form and its transposition

Direct form for downsampling

- Filter : at high speed (input speed)
- $M - 1$ samples out of M are discarded
- the $\boxed{\downarrow M}$ block can be distributed
- multiplication and downsampling can be interchanged
- multiplication is now operated at the low rate
- see figure

Direct transpose form for upsampling

- Similarly the $+$ and $*$ can be sent to the low rate part

- see figure

Polyphase structure for upsampling

- One component is a downsampled version of the sequence with particular phase

- A filter transfer function can be written

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{L-1} h(nL+k)\, z^{-nL-k}$$
$$= \sum_{k=0}^{L-1} z^{-k}\, P_k(z^L) \tag{61}$$

- kth polyphase component $p_k(n)$ and z-transform

$$P_k(z^L) = \sum_{n=-\infty}^{\infty} h(nL+k)\, z^{-nL} = \sum_{n=-\infty}^{\infty} p_k(n)\, z^{-nL} \tag{62}$$

Polyphase structure for upsampling

- Interpolation

$$\begin{aligned} Y(z) &= H(z)\,X(z^L) \\ &= \sum_{k=0}^{L-1} z^{-k}\,P_k(z^L)\,X(z^L) \end{aligned} \tag{63}$$

- $P_k(z^L)\,X(z^L)$ means implement $P_k(z)\,X(z)$ then insert "0"s

- see figure

- L filters operate at the low rate

Polyphase structure for upsampling

- Each branch is in charge of a polyphase component of the output
- Branch i: samples $i, i+L, i+2L, \cdots$
- Implementation as commutator
- Here polyphase components of the first type : counter clock commutator

Polyphase components

- The ideal interpolation filter has cutof π/L

- Each polyphase component is a downsampled version with particular phase

- Hence because of spectrum repetition they are allpass filters with different phases

Polyphase components

- About the spectra of polyphase components

$$\begin{aligned} p_k(n) &= h(nL+k) \\ &= \frac{1}{2\pi}\int_0^{2\pi} P_k(e^{j\Omega'})\, e^{jn\Omega'}\, \mathrm{d}\Omega' \\ &= \frac{1}{2\pi}\int_0^{2\pi} H(e^{j\Omega})\, e^{j(nL+k)\Omega}\, \mathrm{d}\Omega \end{aligned} \tag{64}$$

$$h(nL+k) = \frac{1}{2\pi}\sum_{p=0}^{L-1}\int_{2\pi p/L}^{2\pi(p+1)/L} H(e^{j\Omega})\, e^{j(nL+k)\Omega}\, \mathrm{d}\Omega \tag{65}$$

- With $\Omega'' = \Omega - 2\pi p/L$ and $p' = L - p$

$$h(nL+k) = \frac{1}{2\pi}\int_0^{2\pi/L}\sum_{p'=0}^{L-1} H\left[e^{j(\Omega''-2\pi p'/L)}\right] e^{j(nL+k)(\Omega''-2\pi p'/L)}\, \mathrm{d}\Omega'' \tag{66}$$

Polyphase components

- With $\Omega' = L\Omega$''

$$h(nL+k) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{L}\sum_{p'=0}^{L-1} e^{-j2\pi kp/L}\, H\left[e^{j(\Omega'-2\pi p')/L}\right] e^{j(nL+k)\Omega'}\,\mathrm{d}\Omega' \tag{67}$$

$$P_k(e^{j\Omega}) = \frac{1}{L} e^{jk\Omega/L}\sum_{p=0}^{L-1} e^{-2\pi jkp/L}\, H\left[e^{j(\Omega-2\pi p)/L}\right] \tag{68}$$

- For the ideal interpolation filter

$$P_k(e^{j\Omega}) = e^{jk\Omega/L} \tag{69}$$

Polyphase structure for downsampler

- Can be obtained by transposition of previous one

- see result

- see commutator representation

Polyphase components of second type

- Definition

$$\overline{p}_k(n) = h(nL - k) \tag{70}$$

- Link with type one

$$\begin{aligned}\overline{p}_\rho(n) &= h(nL - \rho) \\ &= h\left[(n-1)L + (L - \rho)\right] \\ &= p_{L-\rho}(n-1)\end{aligned}$$

- see commutator representation for downsampling

- In practice : take care of the polyphase type !

Polyphase components with the ideal filter

- polyphase transfer function

$$P_k(e^{j\Omega}) = e^{jk\Omega/L} \tag{71}$$

- For downsampling (because of different gain)

$$P_k(e^{j\Omega}) = \frac{1}{M}\, e^{jk\Omega/M} \tag{72}$$

- Ideal filter impulse response

$$\tilde{h}(k) = \frac{\sin(\pi k/L)}{\pi k/L} \tag{73}$$

- Hence

$$\tilde{p}_\rho(n) = \frac{\sin[\pi(n+\rho/L)]}{\pi(n+\rho/L)} \tag{74}$$

Polyphase components with the ideal filter

- Original samples are sent to the output and left unchanged

$$\tilde{p}_0(n) = \delta(n) \tag{75}$$

- Only those will be totally correct if an FIR is preferred

- Symmetry properties $\tilde{h}(n) = \tilde{h}(-n)$

$$\begin{aligned} \tilde{p}_\rho(n) &= \tilde{h}(nL + \rho) \\ &= \tilde{h}(-nL - \rho) \\ &= \tilde{h}[(-n-1)L + L - \rho] \\ &= \tilde{p}_{L-\rho}(-n-1) \end{aligned} \tag{76}$$

- see illustration

Polyphase components with the ideal filter

- see illustration

- any FIR design method can be used for the polyphase components