

CALCULATING THE PERFORMANCE OF A SOFT-INFORMATION-BASED BEST LINEAR UNBIASED ESTIMATOR OF AMPLITUDE AND CARRIER PHASE OFFSET

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ABSTRACT

This paper analytically calculates the expectation and the variance of a soft-information-based best linear unbiased estimator of amplitude and carrier phase offset. Long data frames are considered.

The calculation includes the impact on the performance of the presence of training symbols as well as non-gaussianity of the Log-Likelihood Ratios (LLRs) fed to the estimator input. It is also analyzed how the properties of the estimator are affected when the ratio between the mean and variance of the LLRs is not equal to $1/2$.

1. INTRODUCTION

The performance of digital communication systems relies on the availability of accurate estimates of parameters like the amplitude, the carrier phase offset, the propagation delay and, in case of frequency-selective channels, of the channel taps. In systems operating at low signal-to-noise ratios, iterative estimators (see references in [1]) prevent from using a prohibitive number of training symbols. They exploit soft information provided by a turbo receiver. Among them, an expectation-maximization (EM) algorithm-based estimator [1] is a possible solution. However, despite a convergence towards the optimal maximum likelihood estimate ensured under mild conditions, the EM algorithm gives no guarantee about the estimate quality throughout the iterations. Besides, the EM estimator turns out to be strongly biased in some cases especially during the first turbo iterations when the quality of soft information is quite poor.

The goal of this paper is to calculate the expectation and the variance of a soft-information-based best linear unbiased estimator of amplitude and carrier phase offset. Long data frames are considered. The estimator is a good trade-off between complexity and performance. Indeed, from the one hand, it is linear in the observations vector output by

the channel. From the other hand, by design, it delivers, given the available soft information and given this linearity constraint, an unbiased and minimum-variance estimate. The estimator is here restricted to BPSK data modulation. An extension of the analysis to multi-level data modulation and to several-tap channel estimation would not present any major difficulty. It is not done here for the sake of notations clarity.

The paper also explains how to calculate the impact on the performance of the presence of training symbols as well as non-gaussianity of the Log-Likelihood ratios (LLRs) fed to the estimator input. It is also analyzed how the properties of the estimator are affected when the ratio between the mean and variance of the LLRs is not equal to $1/2$.

The sequel of this paper will be organized as follows. The system model will be presented in section 2. Section 3 will give the expression of our estimator. It will also calculate its performance in terms of expectation and variance. Finally, section 4 will compare the estimator performance obtained by simulations with that obtained by our calculations.

2. SYSTEM MODEL

In this section, the transmitter model and the iterative receiver will be successively presented.

2.1. Transmitter model

The transmission scheme is the following. A frame of information bits u_i is encoded by a rate- r convolutional encoder. The resulting encoded bits x_j are interleaved using a random permutation function to give the interleaved coded bits x_k . These bits are then mapped onto BPSK symbols $s_k \in \{+1, -1\}$ according to $s_k = 2x_k - 1$ ($1 \leq k \leq K$ where K is the number of data symbols in a frame). These BPSK symbols are preceded by T training symbols $s_k \in \{+1, -1\}$ with $-T + 1 \leq k \leq 0$ and then transmitted over an AWGN channel. After matched filtering at the receiver, the observations y_k may thus be expressed

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as $(-T + 1 \leq k \leq K)$

$$y_k = w s_k + n_k, \quad (1)$$

where n_k are Gaussian noise samples of variance σ_n^2 . Variable $w = A e^{j\theta}$ denotes the complex amplitude i.e. A is the channel gain and θ is the carrier phase offset.

2.2. Iterative receiver

The receiver is made up with our estimator of complex amplitude w and with a classical turbo receiver like in [2]. The estimator will be presented and analyzed in details in section 3. The turbo receiver includes a soft-in soft-out (SISO) decoder. Extrinsic LLRs on coded bits output by the decoder are fed back to the estimator in order to improve its performance. These LLRs are regarded as a second source of information on the transmitted data symbols at the estimator disposal apart from the channel observations.

3. SOFT-INFORMATION-BASED BEST LINEAR UNBIASED ESTIMATOR OF COMPLEX AMPLITUDE

In this section, we will first give the expression of our soft-information-based best linear unbiased estimator of complex amplitude (subsection 3.1). Secondly, we will calculate its expectation and variance (subsection 3.2).

3.1. Expression of soft-information-based best linear unbiased estimator

As said in the introduction, we restrict ourselves to BPSK data modulation. Let $L^{(n-1)}(x_k)$ ($1 \leq k \leq K$) denote the extrinsic LLRs on coded bits x_k output by the decoder at turbo iteration $n - 1$. These extrinsic LLRs are regarded as random variables with conditional probability density function (pdf) $p(L^{(n-1)}(x_k)|s_k)$. They are commonly approximated as Gaussian random variables [3]:

$$L^{(n-1)}(x_k) = \mu^{(n)} s_k + \nu_k^{(n)}, \quad (2)$$

with mean $\mu^{(n)}$ and variance $\sigma^2(n)$ (i.e. $\nu_k^{(n)} \sim \mathcal{N}(0, \sigma^2(n))$). In this case,

$$p(L^{(n-1)}(x_k)|s_k) = \frac{1}{\sqrt{2\pi\sigma^2(n)}} \exp\left(\frac{-(L^{(n-1)}(x_k) - \mu^{(n)} s_k)^2}{2\sigma^2(n)}\right). \quad (3)$$

Let now also $E_{s|L^{(n-1)},n}\{q\}$ denote the joint expectation of any random variable q with respect to both the symbols s_k given the extrinsic LLRs $L^{(n-1)}(x_k)$ and to channel noise n_k .

In a first step, we will assume that $T = 0$ i.e. no training symbols is used. We want to find a linear estimator which given the available extrinsic LLRs output by the decoder

at each turbo iteration is both unbiased and of minimum variance by design. In mathematical terms, this means that we search for an estimate of w at iteration n , denoted by $\hat{w}^{(n)}$, which is given by $\hat{w}^{(n)} = \sum_{k=1}^K c_k^{(n)} y_k$ where the coefficients $c_k^{(n)}$ are chosen such that $E_{s|L^{(n-1)},n}\{\hat{w}^{(n)}\} = w$ and $E_{s|L^{(n-1)},n}\{|\hat{w}^{(n)}|^2\} - |w|^2$ is minimum. It may be easily shown that

$$\hat{w}^{(n)} = \frac{1}{\sum_{k=1}^K (\eta_k^{(n-1)})^2} \sum_{k=1}^K \eta_k^{(n-1)} y_k, \quad (4)$$

where $\eta_k^{(n-1)}$ is the extrinsic average value of symbol s_k at iteration $n - 1$ calculated from $L^{(n-1)}(x_k)$ as:

$$\eta_k^{(n-1)} = \tanh\left(\frac{L^{(n-1)}(x_k)}{2}\right). \quad (5)$$

In the calculation of coefficients $c_k^{(n)}$, we supposed (2) and $\mu^{(n)}/\sigma^2(n) = 1/2$ ¹. Indeed, since the ratio $\mu^{(n)}/\sigma^2(n)$ is not here assumed to be estimated at the receiver, the estimator considers it to be equal to $1/2$ which is not exactly the case in practice. Subsection 3.2 will take into account in the performance calculation possible non-gaussianity of the extrinsic LLRs as well as $\mu^{(n)}/\sigma^2(n) \neq 1/2$.

3.2. Calculation of the estimator expectation and variance

Let us now calculate $E_{L^{(n-1)},s,n}\{\hat{w}_{EM}^{(n)}\}$, the joint expectation of $\hat{w}^{(n)}$ with respect to the symbols s_k , to a posteriori LLRs $L^{(n-1)}(x_k)$ and to noise n_k . Using (1), (4), the independence of n_k and s_k , assuming that of n_k and $L^{(n-1)}(x_k)$ and supposing long enough frames, it is shown in the appendix that

$$E_{L^{(n-1)},s,n}\{\hat{w}^{(n)}\} \cong \left(\alpha_v^{(n)}/\alpha^{(n)}\right) w. \quad (6)$$

Variable $\alpha_v^{(n)} \triangleq E_{L^{(n-1)},s}\{\eta_k^{(n-1)} s_k\}$ may be calculated as

$$\alpha_v^{(n)} = \int_{-\infty}^{+\infty} \tanh\left(\frac{z}{2}\right) \frac{1}{2} \left[p(z|s_k = +1) - p(z|s_k = -1) \right] dz, \quad (7)$$

whereas $\alpha^{(n)} \triangleq E_{L^{(n-1)},s}\{(\eta_k^{(n-1)})^2\}$ is given by

$$\alpha^{(n)} = \int_{-\infty}^{+\infty} \tanh^2\left(\frac{z}{2}\right) \frac{1}{2} \left[p(z|s_k = +1) + p(z|s_k = -1) \right] dz. \quad (8)$$

Variable z in (7) and (8) is an integration variable replacing $L^{(n-1)}(x_k)$ in order not to encumber the notations. It results from their definitions that both $\alpha_v^{(n)}$ and $\alpha^{(n)}$ are real and $0 \leq \alpha_v^{(n)}, \alpha^{(n)} \leq 1$.

¹which is commonly used [3] although it is not always very accurate

If the Gaussian assumption (2) is made on $L^{(n-1)}(x_k)$, combining (3) and (7) leads to

$$\alpha_v^{(n)} = \int_{-\infty}^{+\infty} \tanh\left(\frac{z}{2}\right) \tanh\left(\frac{\mu^{(n)} z}{\sigma^2(n)}\right) \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2(n)}} \left[\exp\left(\frac{-(z - \mu^{(n)})^2}{2\sigma^2(n)}\right) + \exp\left(\frac{-(z + \mu^{(n)})^2}{2\sigma^2(n)}\right) \right] dz, \quad (9)$$

whereas combining (3) and (8) results in

$$\alpha^{(n)} = \int_{-\infty}^{+\infty} \tanh^2\left(\frac{z}{2}\right) \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2(n)}} \left[\exp\left(\frac{-(z - \mu^{(n)})^2}{2\sigma^2(n)}\right) + \exp\left(\frac{-(z + \mu^{(n)})^2}{2\sigma^2(n)}\right) \right] dz. \quad (10)$$

Thanks to (9) and (10), we notice that $\alpha_v^{(n)} = \alpha^{(n)}$ if the Gaussian assumption on $L^{(n-1)}(x_k)$ is valid and if $\mu^{(n)}/\sigma^2(n) = 1/2$. In this case, by (6), $E_{L^{(n-1)},s,n}\{\hat{w}^{(n)}\} = w$. Otherwise, $\alpha_v^{(n)} \neq \alpha^{(n)}$ and the estimate is biased. This is not surprising since the estimator has been derived precisely assuming the Gaussian assumption on $L^{(n-1)}(x_k)$ and $\mu^{(n)}/\sigma^2(n) = 1/2$.

After the expectation of the estimate, let us now calculate its variance: $\text{var}\{\hat{w}^{(n)}\} = E_{L^{(n-1)},s,n}\{|\hat{w}^{(n)}|^2\} - |E_{L^{(n-1)},s,n}\{\hat{w}^{(n)}\}|^2$. Using again (1), (4), resorting to the same independence assumptions as for the calculation of $E_{L^{(n-1)},s,n}\{\hat{w}^{(n)}\}$ and still assuming long enough frames, the appendix calculates the first term of the variance. The second term is computed thanks to (6). It eventually results that

$$\text{var}\{\hat{w}^{(n)}\} \cong \frac{\sigma_n^2}{K\alpha^{(n)}} + \frac{[\alpha^{(n)} - (\alpha_v^{(n)})^2] |w|^2}{K(\alpha^{(n)})^2}. \quad (11)$$

Let us now take the training symbols into account. Incorporating the T training symbols into the frame, the estimate at iteration n , denoted by $\hat{w}_p^{(n)}$, becomes

$$\hat{w}_p^{(n)} = \frac{1}{T+K} (T\hat{w}_T + K\hat{w}^{(n)}), \quad (12)$$

where $\hat{w}_T \triangleq \frac{1}{T} \sum_{k=-T+1}^0 s_k y_k$ is an unbiased estimator (i.e. $E_{s,n}\{\hat{w}_T\} = w$) with variance $\text{var}\{\hat{w}_T\} = \sigma_n^2/T$ [4]. It is easy to show using (6) and (11) that

$$E_{L^{(n-1)},s,n}\{\hat{w}_p^{(n)}\} \cong \frac{T + (\alpha_v^{(n)}/\alpha^{(n)})K}{K+T} w, \quad (13)$$

$$\text{var}\{\hat{w}_p^{(n)}\} \cong \frac{(T+K/\alpha^{(n)})\sigma_n^2}{(T+K)^2} + \frac{K[\alpha^{(n)} - (\alpha_v^{(n)})^2] |w|^2}{(T+K)^2 (\alpha^{(n)})^2}. \quad (14)$$

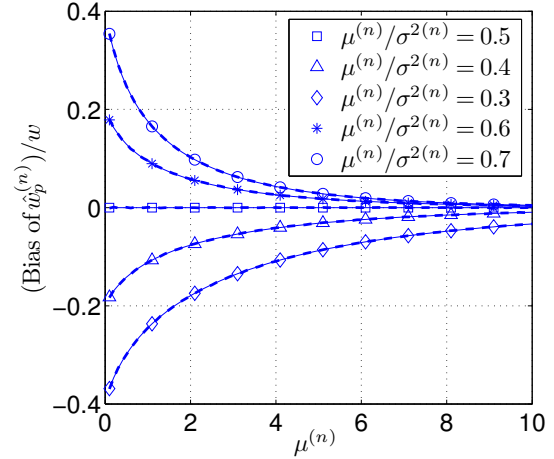


Fig. 1. Normalized bias of $\hat{w}_p^{(n)}$ (i.e. bias of $\hat{w}_p^{(n)}$ divided by w) versus $\mu^{(n)}$ for different values of $\mu^{(n)}/\sigma^2(n)$. Comparison between calculations (solid lines) and simulations (dotted lines). $T = 10$, $K = 1028$, $E_s/N_0 = 4\text{dB}$, $w = 1.0 e^{j45^\circ}$.

4. RESULTS

In this section, we wanted to focus on the estimator behavior independently of the other receiver blocks. So we did not simulate our estimator when embedding it into a turbo receiver. We just imposed at the estimator input perfectly Gaussian LLRs independent of the channel observations. Frames of $K = 1024$ BPSK symbols completed by $T = 10$ training symbols were used. We chose $E_s/N_0 = |w|^2/\sigma_n^2$ equal to 4dB and an actual value of w equal to $1.0 e^{j45^\circ}$.

Fig. 1 shows the bias $b\{\hat{w}_p^{(n)}\}$ of $\hat{w}_p^{(n)}$ divided by w versus $\mu^{(n)}$ for different values of the ratio $\mu^{(n)}/\sigma^2(n)$: 0.3, 0.4, 0.5, 0.6, 0.7. Fig. 2 shows the mean square error (MSE) of $\hat{w}_p^{(n)}$ ($\text{MSE of } \hat{w}_p^{(n)} = \text{var}\{\hat{w}_p^{(n)}\} + |b\{\hat{w}_p^{(n)}\}|^2$) versus $\mu^{(n)}$. Curves giving the MSE of \hat{w}_T (estimator with the training symbols only) and the asymptotical value of the MSE (i.e. when $\mu^{(n)} \rightarrow \infty$) are also represented. This latter curve is also the Cramer Rao bound in data-aided mode [4].

In each figure, the solid curves represent the results obtained by simulations of the estimator behavior whereas the dashed curves are for the results obtained by (13) and (14). In both figures, solid and dashed curves are close to each other and even tend to be indiscernible (especially in fig. 1) which shows the accuracy of our calculations. The remaining gap is due to the long frames approximation. We also observe that the best estimator performance is obtained with $\mu^{(n)}/\sigma^2(n) = 0.5$, i.e. the mean-to-variance ratio for which the estimator has been designed (see subsection 3.1).

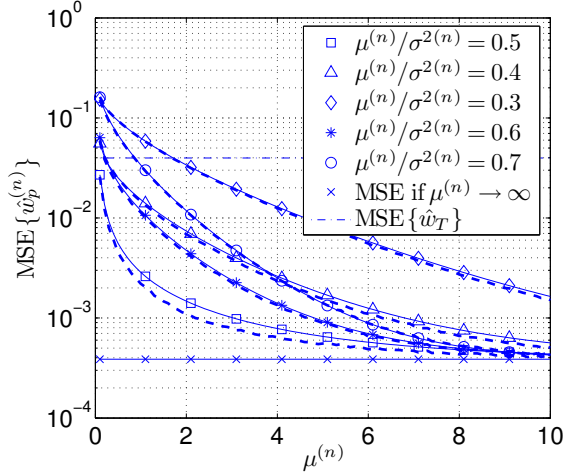


Fig. 2. Mean square error (MSE) of $\hat{w}_p^{(n)}$ versus $\mu^{(n)}$ for different values of $\mu^{(n)}/\sigma^2(n)$. Comparison between calculations (solid lines) and simulations (dotted lines). $T = 10$, $K = 1028$, $E_s/N_0 = 4\text{dB}$, $w = 1.0 e^{j 45^\circ}$.

5. CONCLUSION

This paper analytically calculates the expectation and the variance of a soft-information-based best linear unbiased estimator of amplitude and carrier phase offset. Long data frames are considered.

The presence of training symbols as well as non-gaussianity of the Log-Likelihood Ratios (LLRs) fed to the estimator input are considered in the calculations. The impact of a mean-to-variance ratio of the LLRs not equal to 1/2 was also taken into account and illustrated in the results section.

The spirit of the paper may be extended to the calculation of the expectation and variance of a several-tap channel estimator as well as to multi-level modulation.

6. REFERENCES

- [1] N. Noels, C. Herzet, A. Dejonghe, V. Lottici, H. Steendam, M. Moeneclaey, M. Luise, and L. Vandendorpe, "Turbo-synchronization: an EM algorithm interpretation," in *IEEE International Conference on Communications, ICC'*, 2003, vol. 4, pp. 2933–2937.
- [2] A. Dejonghe and L. Vandendorpe, "Turbo-equalization for multilevel modulation : an efficient low-complexity scheme," in *IEEE International Conference on Communications, ICC'*, New York, USA, May 2002, vol. 3, pp. 1863–1867.
- [3] S. Ten Brink, "Convergence behavior of iteratively decoded parallel concatenated codes," *IEEE Transactions on Communications*, vol. 49, no. 10, pp. 1727–1737, Oct. 2001.
- [4] H. Meyr, M. Moeneclaey, and S. Fetchel, *Digital Communication Receivers : Synchronization, Channel Estimation and Signal Processing*, Wiley Series in Telecommunications and Signal Processing, USA, 1998.

7. APPENDIX

Let us first prove (6), (7) and (8). For long enough frames, we notice that $\hat{\alpha}^{(n)} \triangleq (1/K) \sum_{k=1}^K (\eta_k^{(n-1)})^2$ is a good approximation of $\alpha^{(n)}$ ($= \alpha_v^{(n)}$) if the Gaussian assumption is valid and $\mu^{(n)}/\sigma^2(n) = 1/2$. Thus, we will approximate $\sum_{k=1}^K (\eta_k^{(n-1)})^2$ by $K\alpha^{(n)}$ in the sequel. Using this, (1), (4), the independence of n_k and s_k and assuming that of n_k and $L^{(n-1)}(x_k)$, we may write

$$\begin{aligned} E_{L^{(n-1)},s,n} \{ \hat{w}^{(n)} \} &= \frac{1}{K \alpha^{(n)}} \sum_{k=1}^K \left(E_{L^{(n-1)},s} \{ \eta_k^{(n-1)} s_k \} w \right. \\ &\quad \left. + E_{L^{(n-1)},s} \{ \eta_k^{(n-1)} \} E_n \{ n_k \} \right) \\ &= \left(\alpha_v^{(n)} / \alpha^{(n)} \right) w, \end{aligned}$$

where $\alpha_v^{(n)} \triangleq E_{L^{(n-1)},s} \{ \eta_k^{(n-1)} s_k \}$. This variable is given by

$$\begin{aligned} \alpha_v^{(n)} &= E_s \left\{ E_{L^{(n-1)}|s} \left\{ \tanh \left(\frac{L^{(n-1)}(x_k)}{2} \right) s_k \right\} \right\} \\ &= E_{L^{(n-1)}|s} \left\{ \tanh \left(\frac{L^{(n-1)}(x_k)}{2} \right) \Big|_{s_k=+1} (+1) \right\} \frac{1}{2} \\ &\quad + E_{L^{(n-1)}|s} \left\{ \tanh \left(\frac{L^{(n-1)}(x_k)}{2} \right) \Big|_{s_k=-1} (-1) \right\} \frac{1}{2} \end{aligned}$$

which eventually leads to (7). The factors 1/2 are respectively for a priori probabilities $p(s_k = +1)$ and $p(s_k = -1)$. Variable $\alpha^{(n)} \triangleq E_{L^{(n-1)},s} \{ (\eta_k^{(n-1)})^2 \}$ may be calculated by the kind of same reasoning. It results that

$$\alpha^{(n)} = \int_{-\infty}^{+\infty} \tanh^2 \left(\frac{z}{2} \right) \frac{1}{2} \left[p(z|s_k = +1) + p(z|s_k = -1) \right] dz.$$

We eventually obtained (6), (7) and (8).

Let us now prove (11) still assuming long enough frames.

$$\begin{aligned} E_{L^{(n-1)},s,n} \left\{ \left| \hat{w}^{(n)} \right|^2 \right\} &= \frac{1}{K^2 (\alpha^{(n)})^2} \sum_{k=1}^K \sum_{k'=1}^K \left[E_{L^{(n-1)},s} \left\{ |w|^2 \eta_k^{(n-1)} \eta_{k'}^{(n-1)} s_k s_{k'} \right\} \right. \\ &\quad \left. + E_{L^{(n-1)},s} \left\{ \eta_k^{(n-1)} \eta_{k'}^{(n-1)} \right\} E_n \{ n_k n_{k'}^* \} \right]. \end{aligned}$$

If $k \neq k'$, we have

$$\begin{aligned} E_{L^{(n-1)},s} \left\{ |w|^2 \eta_k^{(n-1)} \eta_{k'}^{(n-1)} s_k s_{k'} \right\} &= |w|^2 E_{L^{(n-1)},s} \left\{ \eta_k^{(n-1)} s_k \right\} E_{L^{(n-1)},s} \left\{ \eta_{k'}^{(n-1)} s_{k'} \right\} \\ &= |w|^2 \alpha_v^{(n)} \alpha_v^{(n)} = \left(\alpha_v^{(n)} \right)^2 |w|^2, \end{aligned}$$

whereas, for $k = k'$,

$$\begin{aligned} E_{L^{(n-1)},s} \left\{ |w|^2 \left(\eta_k^{(n-1)} \right)^2 s_k^2 \right\} &= |w|^2 E_{L^{(n-1)},s} \left\{ \left(\eta_k^{(n-1)} \right)^2 \right\} = \alpha^{(n)} |w|^2. \end{aligned}$$

Since $E_n \{ n_k n_{k'}^* \} = \sigma_n^2 \delta(k - k')$, we have all the elements to calculate $E_{L^{(n-1)},s,n} \{ \left| \hat{w}^{(n)} \right|^2 \}$ and thus $\text{var} \{ \hat{w}^{(n)} \}$ which may be written as (11).