TURBO SYNCHRONIZATION : A COMBINED SUM-PRODUCT AND EXPECTATION-MAXIMIZATION ALGORITHM APPROACH

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ABSTRACT

This paper places turbo synchronization into the sum-product (SP) and the expectation-maximization (EM) algorithm framework. In particular, we show that the combination of these algorithms enables to design low-complexity and very powerfull synchronizers. The proposed synchronizer is compared to a previously-proposed EM framework. As the derivation suggests, our new iterative scheme clearly outperforms the classical use of the EM algorithm.

1. INTRODUCTION

In many communication systems practical implementation of the optimal receiver is far too complex. This has led to an increasing interest in iterative algorithms for approximating joint optimal receivers. In particular, turbo decoding [2] which approximates optimal joint decoding of concatenated codes has been shown to exhibit excellent performance. The so-called turbo principle has then been extended to a number of other receiver functions such as joint demodulation and decoding, joint equalization and decoding... Recently, graphical models have been proposed for the design of iterative algorithms. In particular, factor graphs [3], and its associated *sum-product* (SP) algorithm have been shown to enable a particular clean derivation of iterative algorithms.

In addition to detection and decoding, a receiver has also to perform synchronization i.e. to estimate a number of parameters like the carrier phase, the timing epoch,... In a burst transmission system, most classical synchronizers are based on the maximum-likelihood (ML) criterion and usually work in non data-aided (NDA) mode [4]. In turbo receivers synchronization is a very challenging task. Indeed these systems usually operate at very low signal-to-noise ratios (SNR) and therefore, classical synchronizers may fail to provide reliable enough estimates of the synchronization parameters. In this context an idea, often referred to as turbo synchronization, is to take benefit from the soft information available in turbo receivers to try improving the quality of the estimate delivered by the synchronizer. In particular, a framework based on the expectation-maximization (EM) algorithm [5] has been proposed in [1] for the derivation of iterative synchronizers. This approach is the core of a number of iterative methods proposed so far, see [6] for example.

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In this paper, we consider turbo synchronization throughout the factor-graph and the SP-algorithm framework. Due to the continuous nature of synchronization parameters the direct application of the SP algorithm to the system factor graph results in a prohibitive complexity. An approach to overcome this problem consists in approximating the continuous messages by a Dirac function. Paper [7] for example considers carrier phase synchronization and proposes both a steepest-descent method and a particle-based method to compute the point at which the Dirac function does not cancel out. In this paper, we emphasize that the message relative to synchronization parameters is almost equivalent to a likelihood function built by considering the extrinsic probabilities delivered by the system as a priori information. We therefore propose to compute the representative point of the delta function by means of the EM algorithm. We then show for the case of a turbo-coded transmission that the application of the EM algorithm to the synchronization problem such as proposed in [1] is an approximation of the method proposed in this paper. Performance of the proposed scheme is studied through simulation results for a bit-interleaved coded modulation (BICM) transmission. In particular, we show that the proposed synchronizer enables to outperform classical approaches in terms of accuracy and overall complexity.

2. SYSTEM MODEL

In this paper we consider the synchronization of a linearly data-modulated bandpass signal transmitted through an additive white Gaussian noise (AWGN) channel. The received baseband signal $r(t)$ may then be written as

$$r(t) = A \sum_{k=0}^{K-1} a_k \, u(t - kT - \tau) \, e^{j(2\pi\nu t + \theta)} + w(t), \quad (1)$$

where $u(t)$ is a unit-energy square-root raised-cosine pulse, $w(t)$ is a complex-valued AWGN with baseband power spectral density $2N_0$, T is the symbol period, A denotes the channel gain , τ the symbol timing , ν and θ respectively the carrier frequency and the carrier phase offsets, $a_k \in \mathcal{A}$ the symbol transmitted at time kT and $\mathcal{A}$ the constellation alphabet.

3. MAXIMUM-LIKELIHOOD PARAMETER ESTIMATION

Let $\mathbf{r}$ denote a random vector obtained by expanding the received modulated signal $r(t)$ onto a suitable basis and let $\mathbf{b} = [\nu, \tau, \theta, A]^T$ indicate the vector of synchronization parameters to be estimated from the observation of received vector $\mathbf{r}$. Let also $\mathbf{a}$ denote the vector of transmitted symbols i.e. $\mathbf{a} = [a_0, a_1, \ldots, a_{K-1}]^T$. The problem addressed in this paper is to find the ML estimate $\hat{\mathbf{b}}_{\mathrm{ML}}$ of $\mathbf{b}$ i.e.

$$\hat{\mathbf{b}}_{\mathrm{ML}} = \arg\max_{\tilde{\mathbf{b}}} \{ \ln p(\mathbf{r}|\tilde{\mathbf{b}}) \}, \tag{2}$$

where

$$p(\mathbf{r}|\tilde{\mathbf{b}}) = \sum_{\mathbf{a}} p(\mathbf{r}|\mathbf{a}, \tilde{\mathbf{b}})\, p(\mathbf{a}), \tag{3}$$

and $\tilde{\mathbf{b}}$ is a trial value of $\mathbf{b}$. Unfortunately the ML problem (3) is most of the time intractable. Consequently, rather than computing the exact ML estimate (2) conventional synchronizers proposed in the literature [4] are smart approximations of the true ML solution. In some systems operating at low SNRs these approaches may sometimes fail to provide parameter estimates which are reliable enough. In the next two sections, we present some methods which enable to iteratively compute the exact ML solution (2).

4. SYNCHRONIZATION BASED ON THE EM ALGORITHM

The expectation-maximization (EM) algorithm, first defined in [5], is a method which enables to iteratively solve ML estimation problem. In the particular case of the synchronization of a linearly-modulated signal, it has been shown [1] that the sequence $\{\hat{\mathbf{b}}^n\}_0^\infty$ defined as

$$[\hat{\nu}^n, \hat{\tau}^n] = \arg\max_{\tilde{\nu}, \tilde{\tau}} \left\{ \left| \sum_{k=0}^{K-1} \eta_k^*(\mathbf{r}, \hat{\mathbf{b}}^{n-1})\, y_k(\tilde{\nu}, \tilde{\tau}) \right| \right\} \tag{4}$$

$$\hat{\theta}^n = \arg\left\{ \sum_{k=0}^{K-1} \eta_k^*(\mathbf{r}, \hat{\mathbf{b}}^{n-1})\, y_k(\hat{\nu}^n, \hat{\tau}^n) \right\} \tag{5}$$

$$\hat{A}^n = \frac{|\sum_{k=0}^{K-1} \eta_k^*(\mathbf{r}, \hat{\mathbf{b}}^{n-1})\, y_k(\hat{\nu}^n, \hat{\tau}^n)|}{\sum_{k=0}^{K-1} \rho_k(\mathbf{r}, \hat{\mathbf{b}}^{n-1})}, \tag{6}$$

converges under fairly general conditions to the ML estimate (2); where $\eta_k(\mathbf{r}, \hat{\mathbf{b}}^{n-1})$ and $\rho_k(\mathbf{r}, \hat{\mathbf{b}}^{n-1})$ denote the first and second order a posteriori moments of symbol a_k given current estimate $\hat{\mathbf{b}}^{n-1}$ i.e.

$$\eta_k(\mathbf{r}, \hat{\mathbf{b}}^{n-1}) \triangleq \sum_{a \in \mathcal{A}} a_k\, p(a_k = a|\mathbf{r}, \hat{\mathbf{b}}^{n-1}), \tag{7}$$

$$\rho_k(\mathbf{r}, \hat{\mathbf{b}}^{n-1}) \triangleq \sum_{a \in \mathcal{A}} |a_k|^2 p(a_k = a|\mathbf{r}, \hat{\mathbf{b}}^{n-1}). \tag{8}$$

and

$$y_k(\tilde{\nu}, \tilde{\tau}) \triangleq \int_{-\infty}^{+\infty} r(t)\, e^{-j(2\pi\tilde{\nu}t)}\, u(t - kT - \tilde{\tau})\, dt.$$

We may notice from (7) and (8) that the implementation of an EM-based iterative synchronizer requires the evaluation of marginal a posteriori probabilities $p(a_k|\mathbf{r}, \hat{\mathbf{b}}^{(n-1)})$. These probabilities are unfortunately not available in turbo receivers. A common approach is then to approximate them by using the so-called a posteriori probabilities $\tilde{p}(a_k|\mathbf{r}, \hat{\mathbf{b}}^{(n-1)})$ available at the output of soft-in soft-out (SISO) modules in turbo receivers. Note that this approximation may be rather crude, especially during the first iterations. However, the efficiency of this approach for parameter estimation in turbo systems has already been shown in a number of papers. In the next sections, we give a mathematical justification to this approximation by showing that the EM-based mathematical framework described in this section may actually be seen as an approximation of a more general framework based on both the SP and the EM algorithms.

5. SYNCHRONIZATION VIA COMBINED SP AND EM ALGORITHMS

The sum-product (SP) algorithm, first proposed in [3], is a message-passing algorithm which operates on factor graphs and enables to efficiently compute marginals of the function that the graph represents. In this section we show that the update rules defined by the SP algorithm combined with those of the EM algorithm enable to build very efficient iterative ML estimators.

First, let us observe that the objective function of maximization problem (2) may be written as

$$p(\mathbf{r}|\mathbf{b}) = \sum_{\mathbf{a} \in \mathcal{A}^K} p(\mathbf{r}, \mathbf{a}\,|\,\mathbf{b})$$

i.e. may be regarded as the marginal of a global function $p(\mathbf{r}, \mathbf{a}\,|\,\mathbf{b})$. Marginal probability $p(\mathbf{r}|\mathbf{b})$ may therefore be computed by applying the SP algorithm to the factor graph relative to probability $p(\mathbf{r}, \mathbf{a}\,|\,\mathbf{b})$ (see Fig. 1). Details about the factorization of $p(\mathbf{r}, \mathbf{a}\,|\,\mathbf{b})$ may be found in appendix. If we consider a coded transmission, the dependence between coded bits introduces cycles in the graph represented in Fig. 1. In this case, it can no longer be proved that the results delivered by the SP algorithm are exact. However, empirical results show that the SP algorithm yields very good results

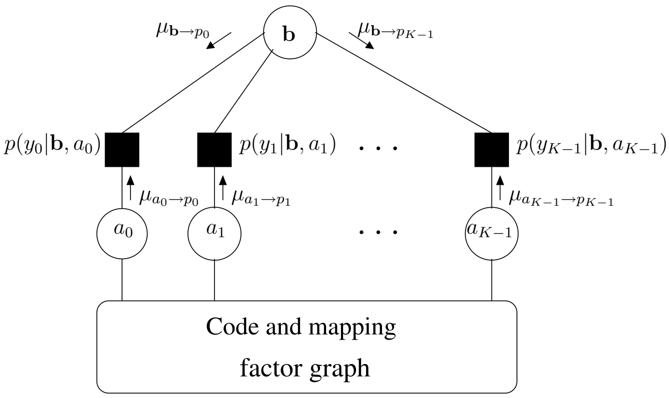

Fig. 1. Factor graph representation of $p(\mathbf{r},\mathbf{a}\,|\,\mathbf{b})$. Factor nodes and variable nodes are respectively denoted by squares and circles.

even when the graph has cycles. Another consequence of the presence of cycles in the factor graph is that the application of the SP algorithm leads to an iterative algorithm with no natural termination. We are therefore required to define a message-passing schedule in order to specify the messages which are updated at each step of the algorithm. Denoting by $\mu^m_{a_k\to p_k}(a_k)$ (resp. $\mu^m_{\mathbf{b}\to p_k}(\mathbf{b})$) the message passing from variable node a_k (resp. $\mathbf{b}$) to factor node $p(y_k|\mathbf{b},a_k)$ at iteration m, we define the following message passing schedule: at each iteration new messages $\mu^{m+1}_{\mathbf{b}\to p_k}(\mathbf{b})$ are first computed by taking into account messages $\mu^m_{a_k\to p_k}(a_k)$; messages $\mu^{m+1}_{a_k\to p_k}(a_k)$ are then updated by applying the SP algorithm on the lower part of the factor graph in Fig. 1 i.e. by exploiting the code structure underlying transmitted symbols $\mathbf{a}$. Using SP algorithm update rules we then have

$$\mu^{m+1}_{\mathbf{b}\to p_k}(\mathbf{b}) \sim \sum_{\mathbf{a}}\prod_{l\neq k} p(y_l|\mathbf{b},a_l)\;\mu^m_{a_l\to p_l}(a_l), \tag{9}$$

where $\sim$ denotes equality up to a multiplicative normalization factor. Note that since the number of messages arriving at variable node $\mathbf{b}$ is large, (9) may be approximated by

$$\mu^{m+1}_{\mathbf{b}\to p_k}(\mathbf{b}) \sim \sum_{\mathbf{a}}\prod_{l} p(y_l|\mathbf{b},a_l)\;\mu^m_{a_l\to p_l}(a_l). \tag{10}$$

i.e. by the product of all the messages arriving at node $\mathbf{b}$. Making this approximation, messages $\mu^m_{\mathbf{b}\to p_k}(\mathbf{b})$ no longer depend on index k and may therefore simply be denoted by $\mu^m_{\mathbf{b}\to p}(\mathbf{b})$. Note also that since $\mathbf{b}$ is a continuous-valued variable, messages $\mu^m_{\mathbf{b}\to p}(\mathbf{b})$ are arbitrary density functions and are therefore impractical to handle. An approach to overcome this problem consists in approximating outgoing messages of continuous variables by parameterized canonical distributions. In particular, it may be shown [8] that if the number of observations is large in comparison to the dimensionality of $\mathbf{b}$, probability density function $\mu^m_{\mathbf{b}\to p}(\mathbf{b})$ may be well-approximated [1] by

$$\mu^m_{\mathbf{b}\to p}(\mathbf{b}) \simeq \delta(\mathbf{b}-\hat{\mathbf{b}}^m) \tag{11}$$

i.e. density $\mu^m_{\mathbf{b}\to p}(\mathbf{b})$ may be assumed to be concentrated in a neighborhood of a given point $\hat{\mathbf{b}}^m$. A common approach to compute $\hat{\mathbf{b}}^m$ is to choose the point at which the probability density is maximized i.e.

$$\hat{\mathbf{b}}^m = \arg\max_{\tilde{\mathbf{b}}} \mu^m_{\mathbf{b}\to p}(\tilde{\mathbf{b}}). \tag{12}$$

Note that (10) may also be rewritten as

$$\mu^{m+1}_{\mathbf{b}\to p}(\mathbf{b}) = \sum_{\mathbf{a}} p(\mathbf{r}|\mathbf{b},\mathbf{a})\,p^m(\mathbf{a}), \tag{13}$$

where $p^m(\mathbf{a}) \triangleq \prod_l \mu^m_{a_l\to p_l}(a_l)$. Maximization problem (12) may therefore be regarded as an intermediate ML problem in which the objective function is computed according to (13) i.e. by using messages $\mu^m_{a_k\to p_k}(a_k)$ as a priori information on the transmitted symbols.

As mentioned in section 4, maximum-likelihood problems can be solved efficiently by means of the EM algorithm. We therefore propose to use the EM algorithm at each iteration of the SP algorithm to compute estimate $\hat{\mathbf{b}}^m$. Due to the similarity of (3) and (13), application of the EM algorithm to the ML problem (12) leads to the same update parameter expressions as those defined in (4), (5), (6). However, due to the particular factorization of the a priori probability $p^m(\mathbf{a})$, the required a posteriori probabilities may be computed very easily as

$$p^m(a_k|\mathbf{r},\hat{\mathbf{b}}^{m,n}) \sim p(y_k|a_k,\hat{\mathbf{b}}^{m,n})\,\mu^m_{a_k\to p_k}(a_k), \tag{14}$$

where the notation $p^m(\cdot)$ denotes the fact that messages $\mu^m_{a_k\to p_k}(a_k)$ are used as a priori probabilities and $\hat{\mathbf{b}}^{m,n}$ represents the estimate generated at the n^{th} iteration of the EM algorithm and at the m^{th} iteration of the SP algorithm. The maximization of (13) via the EM algorithm exhibits therefore a very low complexity since both the computation of (4)-(6) (i.e. M-step) and the computation of the a posteriori probabilities $p^m(a_k|\mathbf{r},\hat{\mathbf{b}}^{m,n})$ (i.e. E-step) are straightforward.

Assuming then that distribution $\mu^{m+1}_{\mathbf{b}\to p}(\mathbf{b})$ tends to actual probability $p(\mathbf{r}|\mathbf{b})$ after the SP algorithm convergence and that the EM algorithm convergence conditions are fulfilled, the successive computations of $\hat{\mathbf{b}}^{m,n}$ leads to a sequence of estimates which converges to the ML solution (2). Note that the derivation of the proposed SP-EM estimator does not depend on the modulation and the code. The framework presented in this section is therefore very general and may be properly extended to the estimation of a wide range of parameters in various transmission schemes.

[1] We assume that the function is unimodal. This may be ensured for example by transmitting some pilots.

$\hat{\mathbf{b}}^0 = \mathbf{b}_0$ (initialization);
for $n = 1 \rightarrow N$ **do**
    Computation of $p(a_k|\mathbf{r}, \hat{\mathbf{b}}^{n-1})$ (or $\tilde{p}(a_k|\mathbf{r}, \hat{\mathbf{b}}^{n-1})$) $\forall\, a_k$;
    Computation of $\hat{\mathbf{b}}^n$ according to (4)-(6);
end

Fig. 2: Summary of the operations performed by the EM synchronizer.

6. RELATIONSHIP BETWEEN THE EM AND SP-EM APPROACHES

In the sequel, to avoid confusion, we will refer to the EM algorithm implemented at each iteration of the SP algorithm as EM_{SP} whereas the EM synchronizer presented in section 4 will simply be denoted by EM. The operations performed by the EM and SP-EM synchronizers are summarized in Fig. 2 and Fig. 3.

As mentioned above, a posteriori probabilities $p(a_k|\mathbf{r}, \hat{\mathbf{b}}^{n-1})$ required to implement the EM synchronizer are not available in turbo receivers. A common approach to deal with this problem is to approximate these probabilities by the so-called a posteriori probabilities $\tilde{p}(a_k|\mathbf{r}, \hat{\mathbf{b}}^{n-1})$ delivered at each turbo iteration by the SISO modules which make up the turbo receiver[2]. The EM synchronizer is therefore an approximation of an actual EM algorithm.

The SP-EM synchronizer is an approximation of the SP algorithm: the message related to synchronization parameters is approximated by a delta function and the non-zero value of the delta function is computed thanks to an EM algorithm. The SP-EM and the EM synchronization derive therefore from fundamentally different approaches. We may however find a connection between them. Indeed, using turbo-code factor graph, messages $\mu^m_{a_k \rightarrow p_k}(a_k)$ required by the SP-EM synchronizer may be shown [3] to be related to extrinsic probabilities $p^m_{e_1}(a_k)$ and $p^m_{e_2}(a_k)$ delivered by the two SISO decoders at iteration m as

$$\mu^m_{a_k \rightarrow p_k} = p^m_{e_1}(a_k)\, p^m_{e_2}(a_k) \quad \text{for systematic bits} \tag{15}$$

$$= p^m_{e_i}(a_k) \quad \text{for parity bits from encoder } i. \tag{16}$$

Notice to avoid confusion that, due to the particular message-passing schedule chosen in section 5, one turbo iteration corresponds to one SP algorithm iteration. Note also that extrinsic probabilities $p^m_{e_1}(a_k)$ and $p^m_{e_2}(a_k)$ are related [2] to the pseudo a posteriori probabilities $\tilde{p}(a_k|\mathbf{r}, \hat{\mathbf{b}}_{\mathbf{old}})$ according to

$$\tilde{p}(a_k|\mathbf{r}, \hat{\mathbf{b}}_{\mathbf{old}}) \sim p(y_k|a_k, \hat{\mathbf{b}}_{\mathbf{old}})\, p^m_{e_1}(a_k)\, p^m_{e_2}(a_k) \quad \text{for systematic bits} \tag{17}$$

[2]In order to reduce the approximation made on posterior probabilities $p(a_k|\mathbf{r}, \hat{\mathbf{b}}^{n-1})$, some authors [1], [9] propose to perform several turbo iterations before delivering pseudo a posteriori probabilities $\tilde{p}(a_k|\mathbf{r}, \hat{\mathbf{b}}^{n-1})$. We do not consider this case in the sequel.

$\hat{\mathbf{b}}^{0,0} = \mathbf{b}_0$ (initialization);
for $m = 1 \rightarrow M$ (SP iterations) **do**
    $\hat{\mathbf{b}}^{m,0} = \hat{\mathbf{b}}^{m-1,N}$;
    Computation of $\mu^m_{a_l \rightarrow p_l}(a_l)$;
    for $n = 1 \rightarrow N$ (EM_{SP} iterations) **do**
        Computation of $p^m(a_k|\mathbf{r}, \hat{\mathbf{b}}^{(m,n-1)})$ $\forall\, a_k$;
        Computation of $\hat{\mathbf{b}}^{(m,n)}$ according to (4)-(6);
    end
end

Fig. 3: Summary of the operations performed by the SP-EM synchronizer.

$$\sim p(y_k|a_k, \hat{\mathbf{b}}_{\mathbf{old}})\, p^m_{e_i}(a_k) \quad \text{for parity bits from encoder } i \tag{18}$$

where $\hat{\mathbf{b}}_{\mathbf{old}}$ denotes an estimate computed at the previous turbo iteration. Comparing then (17) (resp. (18)) to (14) by taking (15) (resp. (16)) into account, we may notice that the EM approach is an approximation of the SP-EM synchronizer in which only one EM_{SP} iteration is performed at each SP iteration. Let us however insist on the fact that the SP-EM synchronizer is not a trivial extension of the EM synchronizer i.e. the iterative nature of the SP-EM algorithm (at a given turbo iteration) cannot be justified by standard EM arguments [1].

7. SIMULATION RESULTS

In this section we study the performance of the SP-EM synchronizer through simulation results in the case of a BICM transmission. The transmitter is then made up with a binary encoder and a constellation mapper separated by a bit interleaver. At the receiver a turbo-demodulation scheme is considered. We consider a rate-$\frac{1}{2}$ non-systematic convolutional encoder with generators $(g_1, g_2) = (37, 21)_8$ and use 8-PSK modulation with set partitioning mapping. All synchronization parameters but the carrier phase are assumed to be known. Figure 4 represents the phase estimation mean square error (MSE) and BER versus E_b/N_0-ratios achieved by the system when it is synchronized by either the SP-EM algorithm or the conventional EM method or a NDA Viterbi&Viterbi (VV) synchronizer [4]. The modified Cramer-Rao bound (MCRB) and the BER achieved by a perfectly-synchronized system are plotted with dashed lines. The iterative synchronizers are all initialized using an estimate computed with the Viterbi&Viterbi phase estimator. At each turbo iteration, the SP-EM approach performs 15 EM_{SP} iterations. We assume that the phase ambiguity problem is perfectly solved so that, at each iteration, $|\theta - \hat{\theta}| < \pi/8$ radians. The phase offset is uniformely distributed on $[-\pi/8, \pi/8]$. Transmitted frames are made up with 1000 8-PSK symbols and 15 turbo iterations are performed. We see from Fig. 4 that the SP-EM synchronizer outperforms classical approaches. In particular it enables to recover the BER of a perfectly-synchronized system. Fig. 5 illustrates the speed of convergence of the system

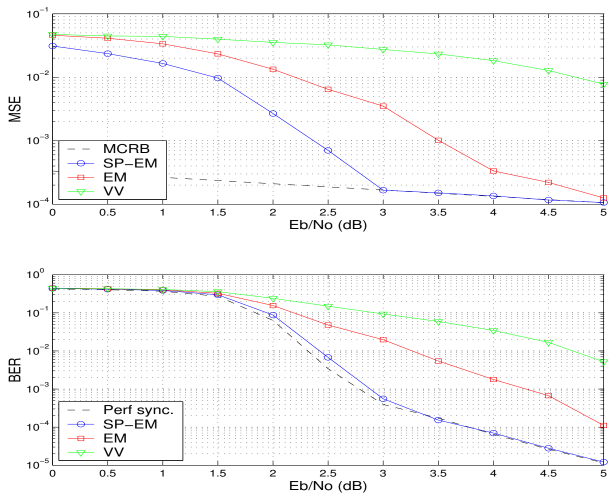

Fig. 4. MSE and BER vs. the E_b/N_0-ratio at the 15[th] turbo iteration.

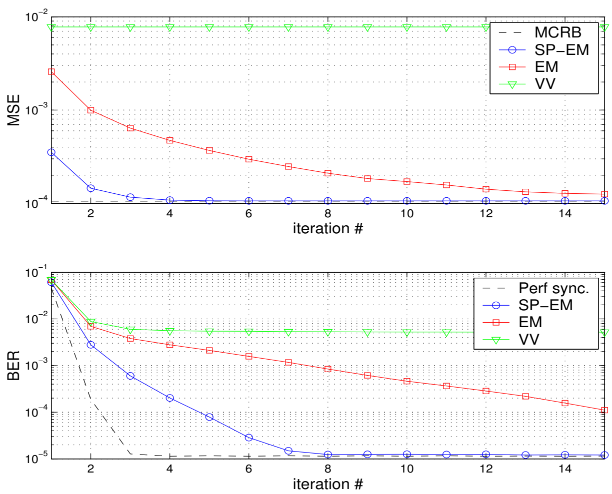

Fig. 5. MSE and BER vs. the number of performed turbo iterations for $E_b/N_0 = 5$dB

in terms of MSE and BER for $E_b/N_0 = 5$dB. Let us insist on the fact that the $\mathrm{EM}_{\mathrm{SP}}$ iterations performed at each turbo (or equivalently SP) iteration have a very low complexity with respect to the complexity of one turbo iteration. Comparing then the speed of convergence of the EM and the SP-EM synchronizers, we see therefore that the SP-EM synchronizer, which exhibits a high speed of convergence in terms of turbo iterations, leads to a receiver which has a much lower overall complexity than the one based on the EM synchronizer.

8. CONCLUSION

In this paper, we apply the SP and the EM algorithms to the issue of digital receiver synchronization. The SP message outgoing synchronization variable node is approximated by a delta function and the non-zero value of the delta function is computed by means of the EM algorithm. At a given turbo iteration, the iterative nature of the proposed synchronizer is then fully justified by our framework. The standard EM approach [1] fails to derive such an algorithm resulting in poorer performance, both in terms of accuracy and speed of convergence.

9. APPENDIX

Using Bayes rule and the fact that $\mathbf{a}$ and $\mathbf{b}$ are independent, we have

$$p(\mathbf{r}, \mathbf{a}|\mathbf{b}) = p(\mathbf{r}|\mathbf{a}, \mathbf{b})\, p(\mathbf{a}). \tag{19}$$

Noticing then that the matched-filter outputs $\mathbf{y}$ are sufficient statitics of the considered estimation problem we have

$$p(\mathbf{r}, \mathbf{a}|\mathbf{b}) = p(\mathbf{y}|\mathbf{a}, \mathbf{b})\, p(\mathbf{a}). \tag{20}$$

Finally, taking into account that the noise which affects the observations is white, we get

$$p(\mathbf{r}, \mathbf{a}|\mathbf{b}) = \prod_k p(y_k|a_k, \mathbf{b})\, p(\mathbf{a}). \tag{21}$$

Note that probability $p(\mathbf{a})$ may be further factorized according to the code and the mapping which are considered for the transmission. Nevertheless, for the sake of generality we do not explicitly represent the factor graph relative to $p(\mathbf{a})$ in Fig. 1.

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